THE APPLICATION OF VON MISES' VARIABLES TO THE PROBLEM OF LAMINAR BOUNDARY LAYER PROPAGATION ALONG A WALL

(PRIMENENIE PEREMENNYKH MIZESA K ZADACHE O RAPROSTRANENII LAMINARNOI STRUI VDOL' STENKI)

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1. The general problem. Let us imagine we have a semi-infinite plate at the leading edge of which there is an infinitely thin slot source, from which there issues a stream of fluid, identical to that which fills the space surrounding the plate; the fluid then flows along one of the sides of the plate. We assume that the fluid in the surrounding space travels at constant velocity u_0 = const in the same direction as the stream and so forms a neighbouring flow.

We locate the origin O of our system of rectangular coordinates at the source of the stream, Ox being directed along the plate in the direction of the stream. We will assume that the region where the motion takes place is a boundary layer, the pressure gradient of which is zero. To solve the problem we make use of the boundary layer equation in the form proposed by von Mises [1].

$$\frac{\partial u}{\partial \xi} = \nu \frac{\partial}{\partial \Psi}\left(u \frac{\partial u}{\partial \Psi}\right) \qquad \left(\xi = x, \ldots, \Psi = \int_0^y u\,dy\right) \tag{1.1}$$

where Ψ is the stream function and u is the velocity component on the x axis.

The solution to (1.1) should satisfy boundary conditions of the following form:

$$u = 0 \quad \text{for } \Psi = 0, \qquad u = u_0 \quad \text{for } \Psi = \Psi_\infty \tag{1.2}$$

where the volume-flow through any stream section $\Psi_\infty = \infty$ when $u_0 \neq 0$.

If $u_0 = 0$, i.e. we are solving the problem of the propagation of the stream without the neighbouring flow [2,3], Ψ_∞ has a finite value. In addition to having the boundary conditions (1.2) it is then essential to have the additional condition of the quantitative value of the stream strength. We can easily obtain this by rewriting Equation (1.1) in the following form:

$$\frac{\partial (u_0 - u)}{\partial \xi} = \nu \frac{\partial}{\partial \Psi}\left[u \frac{\partial (u_0 - u)}{\partial \Psi}\right] \tag{1.3}$$

and then after multiplying Equation (1.3) by $\Psi d\Psi$ we can integrate with respect to Ψ and ξ, taking care to satisfy boundary conditions (1.2). This results in an integral expression of the following form:

$$\frac{\nu u_0^2}{2}\xi - \int\limits_0^{\Psi_\infty} (u_0 - u)\,\Psi d\Psi = E = \text{const} \tag{1.4}$$

When solving the stream problem with the neighbouring flow it is necessary to see that the solution to Equation (1.1) satisfies boundary conditions (1.2) and the integral condition (1.4) for a given condition $E = E_0$.

2. Propagation of the stream without the neighbouring flow. The solution to the $u_0 = 0$ problem must satisfy Equation (2.1), boundary conditions (1.2) and the integral condition (1.4), which, for $u_0 = 0$ takes this form:

$$\int\limits_0^{\Psi_\infty} u\Psi d\Psi = E_0 \tag{2.1}$$

Bearing in mind that Ψ is the stream function it is easy to see that condition (2.1) corresponds to the integral conditions already derived in [2,3].

Let us try and find a solution to Equation (1.1) in this form:

$$u = \sqrt{\frac{E_0}{\nu\xi}}\,\varphi(\eta_1), \qquad \eta_1 = \Psi (E_0 \nu \xi)^{-1/4} \tag{2.2}$$

In order to find function ϕ we obtain a differential equation

$$2(\varphi^2)'' + \eta_1 \varphi' + 2\varphi = 0 \tag{2.3}$$

This should be solved with the following boundary conditions:

$$\varphi = 0 \quad \text{at} \quad \eta_1 = 0, \quad \varphi = 0 \quad \text{at} \quad \eta_1 = \eta_\infty \tag{2.4}$$

and integral condition (2.1) should be satisfied.

A particular solution to Equation (2.3) which automatically satisfies the first boundary condition (2.4) can be found in implicit form, and after satisfying the second boundary condition in (2.4) it appears as

$$\varphi = \frac{1}{6}\eta_\infty^2(\sqrt{t} - t^2) \qquad (t = \eta_1 / \eta_\infty) \tag{2.5}$$

On satisfying integral condition (2.1) we find that $\eta_\infty = 2.515$ and the solution to Equation (2.3) finally takes the form

$$\varphi = 1.054(\sqrt{t} - t^2) \tag{2.6}$$

Remembering that Ψ is the stream function, and therefore

$$y = \int_0^\Psi \frac{d\Psi}{u} \tag{2.7}$$

we can transform from von Mises' variables to variables (x, y). If we insert expression (2.6) into formula (2.7) and express Ψ in terms of η_1 we can find the relation between parameter t and variables (x, y) in the form

$$\frac{2}{\eta_\infty}\ln\frac{1+\sqrt{t}+t}{(1-\sqrt{t})^2} + \frac{12}{\sqrt{3}\eta_\infty}\operatorname{arc\,tg}\frac{\sqrt{3t}}{\sqrt{t}+2} = y\sqrt[4]{\frac{E_0}{\nu^3 x^3}} \tag{2.8}$$

If we eliminate parameter t from (2.6) and (2.8) we can find the dimensionless longitudinal velocity profile (Fig. 1) in the physical plane where

$$\varphi(t) = F(\zeta_1) \qquad \left(\zeta_1 = y\sqrt[4]{\frac{E_0}{\nu^3 x^3}}\right)$$

We thus have a formula for the fluid velocity in the stream

$$u = \sqrt{\frac{E_0}{\nu x}}\, F\left(y\sqrt{\frac{E_0}{\nu^3 x^3}}\right) \tag{2.9}$$

The frictional stress at the wall is

$$\tau_w = 0.221\,\mu\sqrt[4]{\frac{E_0^3}{\nu^5 x^5}}\ldots \tag{2.10}$$

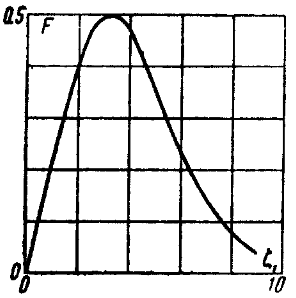

FIG. 1.

3. Asymptotic solution of the problem of stream propagation with a neighbouring flow. It is only possible to arrive at a solution for $u_0 \neq 0$ for high values of ξ because of the expansion in series. For convenience, instead of inserting quantity E_0 into expression (1.4) we introduce another quantity which has the dimension of length

$$L_* = \frac{E_0}{\nu u_0^2} \tag{3.1}$$

Then the integral expression (1.4) takes this form:

$$\int_0^{\Psi_\infty=\infty} (u_0 - u)\, \Psi d\Psi = \nu u_0^2 \left(\frac{1}{2}\xi - L_0\right) \tag{3.2}$$

We will look for a series solution to Equation (1.1), thus

$$\frac{u}{u_0} = f_0(\eta_2) + \frac{f_1(\eta_2)}{(\xi / L_0)} + \frac{f_2(\eta_2)}{(\xi / L_0)^2} + \ldots \qquad \left(\eta_2 = \frac{\Psi}{\sqrt{\nu u_0 (\xi + L_0)}}\right) \tag{3.3}$$

If we insert expansion (3.3) into Equation (1.1) we arrive at a system of differential equations for determining the coefficients in expansion (3.3) in the form

$$\begin{aligned} &(f_0^2)'' + \eta_2 f_0' = 0 \\ &(f_0 f_1'') + \frac{1}{2}\eta_2 f_1' + f_1 = 0 \\ &\left(f_0 f_2 + \frac{1}{2} f_1^2\right)'' + \frac{1}{2}\eta_2 f_2' + 2f_2 + f_1 = 0 \end{aligned} \tag{3.4}$$

The corresponding boundary conditions which are obtained by putting expansion (3.3) into the boundary conditions (2.2) appear as follows:

$$f_0 = f_1 = f_2 = \ldots = 0 \quad \text{при } \eta_2 = 0, \qquad f_0 = 1,\ f_1 = f_2 = \ldots = 0 \quad \text{при } \eta_2 = \infty \tag{3.5}$$

The solutions to the differential equations (3.4) should also satisfy the system of integral conditions

$$\int_0^\infty (1 - f_0)\, \eta_2 d\eta_2 = \frac{1}{2}, \qquad \int_0^\infty f_1 \eta_2 d\eta_2 = \frac{3}{2}, \qquad \int_0^\infty f_2 \eta_2 d\eta_2 = -\frac{3}{2} \tag{3.6}$$

We should note that a long way downstream from the source, because of the decreased intensity of the stream its flow close to the plate will, in fact, be that of a boundary layer on a plate in a longitudinal constant velocity stream. It follows that the solution of (3.3) for $\xi \to \infty$

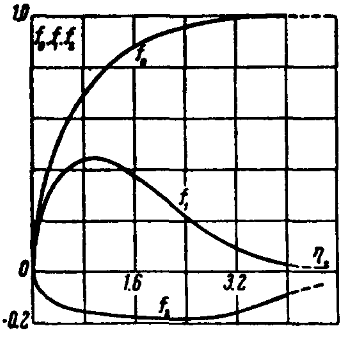

FIG. 2.

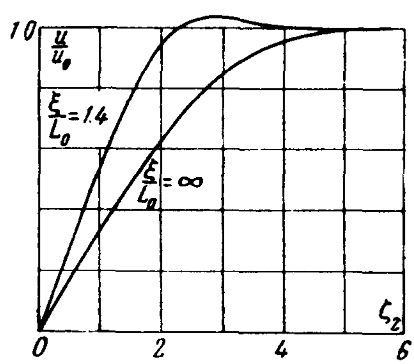

FIG. 3.

should tend to the Blasius solution [4]. It is easy to see that solution (3.3) really does satisfy this condition, because the differential equation and the boundary conditions from which f_0 the first term in the expansion (3.3) is determined correspond exactly to the equation and the boundary conditions of the Blasius problem re-calculated in von Mises' variables. The solution to the Blasius equations is tabulated and it is only necessary to re-calculate it in von Mises' variables. Numerical checking reveals that this solution automatically satisfies the first integral condition of system (3.6).

Functions $f_1(\eta_2)$ and $f_2(\eta_2)$, which represent the additional velocity due to the effect of the stream, were found by numerical integration of the corresponding differential equations in system (3.4), satisfying, at the same time, boundary conditions (3.5) and integral conditions (3.6). Graphs of functions f_0, f_1, f_2 are shown in Fig. 2.

Returning now to the physical (x, y) plane, using the expression

$$\zeta_2 = y\sqrt{\frac{u_0}{\nu(x+L_0)}} = \int_0^{\eta_2} \frac{d\eta_2}{f_0 + (L_0/\xi)f_1 + (L_0^2/\xi^2)f_2 + \dots} \qquad (3.7)$$

it is possible to find the velocity distribution u/u_0. In Fig. 3 we show a velocity distribution curve for the physical plane for $\xi/L_0 = 1.40$ and, for comparison, we also show a curve of velocity distribution for $\xi/L_0 = \infty$, i.e. the Blasius distribution.

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